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THE METHOD OF ANALYZING POTENTIAL ACCURACY OF MULTI-COORDINATE RADAR MEASUREMENTS

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The article sets out an approach making it possible to automate the process of analytical estimation of the accuracy of measuring the parameters of *M* signals in a radar having a digital antenna array.

One of the most important problems of modern radiolocation is the analytical estimation of limiting attainable accuracy of multi-coordinate measurements in multi-signal situations. Its absence compels developers to analyze accuracy properties of competing methods of processing by static modeling, but this implies substantial computational costs. Therefore, the objective of the article is development of the method of forming the Cramer–Rao bound making it possible to assess potential attainable accuracy of measurements in multi-coordinate radars using digital antenna arrays (DAA).

The reference point of the assumed approach is representation of the analytical model of the multi-coordinate radar system by means of the end products of the matrices [1]. When deriving the Cramer–Rao bound in the case of M sources it is the only way to use known findings corresponding to a one-signal mono-parametric reception situation. This method substantially simplifies the solution of the problem considered although the main difficulty related to the great dimension of the results of matrix differentiation still remains.

Therefore further development of the apparatus of matrix end multiplications in order to automate the formation process of their multi-dimensional derivative is of interest, as well as to reduce the dimension of computations when deriving the final record of the Fisher information matrix, used as a model to construct the accuracy lower bound.

The algorithm of such type of forming the matrix derivative will be conveniently considered using the example of a four-coordinate matrix of a pulse radar with DAA [1], when angular substrates together with radial speeds ands ranges of M sources are subject to measurement:

$$P = Q \blacksquare F \blacksquare S \blacksquare D, \qquad (1)$$

where Q and F are $R \times M$ matrices, S and D have, respectively, dimensions $T \times M$ and $N \times M$, while the symbol $\blacksquare$ means the transposition operation of the end product (TEP), which for the $g \times p$ matrix $A = [a_{ij}]$ and the $s \times p$ matrix B, represented as the block matrix $[B_j]$ ($B = [B_j]$, $j = 1,\ldots, p$), is determined by the equality [1]

$$A \blacksquare B = [a_{ij} \cdot B_j].$$

For instance, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$, then

$$A \blacksquare B = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & a_{12}\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} & a_{13}\begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \\ \hdashline a_{21}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & a_{22}\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} & a_{23}\begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \end{bmatrix}.$$

By analogy with [2] as applicable to a flat equidistant DAA, the matrices Q and F may be treated as $R \times M$ matrices of the values for the characteristics of directivity R of the receiving channels in directions of M sources in the azimuthal and elevation planes, respectively, S as $T \times M$ matrix of responses T of synthesized frequency filters on frequencies of M signals, while D as a similar $N \times M$ matrix of signal characteristics N of range strobes. In this case in the absence of noises the set of output voltages of DAA receiving channels may be written in the form $U = P \cdot A$, where $A = [\dot{a}_1 \; \dot{a}_2 ... \dot{a}_M]^T$ is the vector of complex amplitudes of signals.

Under the influence of two sources of signals, for instance,

$$Q = \begin{bmatrix} Q_1(q_1) & Q_1(q_2) \\ Q_2(q_1) & Q_2(q_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Q_R(q_1) & Q_R(q_2) \end{bmatrix}, F = \begin{bmatrix} F_1(x_1) & F_1(x_2) \\ F_2(x_1) & F_2(x_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_R(x_1) & F_R(x_2) \end{bmatrix}, S = \begin{bmatrix} S_1(\omega_1) & S_1(\omega_2) \\ S_2(\omega_1) & S_2(\omega_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ S_T(\omega_1) & S_T(\omega_2) \end{bmatrix},$$

$$D = \begin{bmatrix} D_1(d_1) & D_1(d_2) \\ D_2(d_1) & D_2(d_2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ D_N(d_1) & D_N(d_2) \end{bmatrix}, P = Q \blacksquare F \blacksquare S \blacksquare D =$$

$$= \left[\begin{array}{c|c} Q_1(q_1)\begin{bmatrix} F_1(x_1) \\ \cdot \\ F_R(x_1) \end{bmatrix} & Q_1(q_2)\begin{bmatrix} F_1(x_2) \\ \cdot \\ F_R(x_2) \end{bmatrix} \\ \hline \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hline Q_R(q_1)\begin{bmatrix} F_1(x_1) \\ \cdot \\ F_R(x_1) \end{bmatrix} & Q_R(q_2)\begin{bmatrix} F_1(x_2) \\ \cdot \\ F_R(x_2) \end{bmatrix} \end{array} \right] \blacksquare \left[\begin{array}{c|c} S_1(\omega_1)\begin{bmatrix} D_1(d_1) \\ \cdot \\ D_N(d_1) \end{bmatrix} & S_1(\omega_2)\begin{bmatrix} D_1(\omega_2) \\ \cdot \\ D_N(d_2) \end{bmatrix} \\ \hline \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hline S_T(\omega_1)\begin{bmatrix} D_1(d_1) \\ \cdot \\ D_N(d_1) \end{bmatrix} & S_T(\omega_2)\begin{bmatrix} D_1(d_2) \\ \cdot \\ D_N(d_2) \end{bmatrix} \end{array} \right].$$

In this case the Cramer–Rao lower bound according to [2] will be written as

$$J = 2\sigma^2 \left[\begin{array}{c:c} P^T \cdot P & (A^* \otimes P^T) \cdot \dfrac{\partial P}{\partial Y} \\ \hdashline \left(\dfrac{\partial P}{\partial Y}\right)^T \cdot (A \otimes P) & \left(\dfrac{\partial P}{\partial Y}\right)^T \cdot (A A^* \otimes 1_{RRTN}) \cdot \dfrac{\partial P}{\partial Y} \end{array} \right]^{-1}, \tag{2}$$

where $\partial P / \partial Y$ is the Noidekker derivative from the matrix P by the vector Y whose computation is a stumbling block (Y is composed from the unknown parameters of signals M of the sources); 1_{RRTM} is the identity matrix of dimension $R \times R \times T \times N$; $\otimes$ is the symbol of the Kronecker product.

As it turned out, the process of finding $\partial P / \partial Y$ easily lends itself to automation using a PC. In this case, elements of co-factors forming the block matrix P should be used as initial data.

The factorization property of the Noidekker derivative of TEP, which in the given case will be written as:

$$\frac{\partial P}{\partial Y} = \frac{\partial}{\partial Y}(Q \blacksquare F \blacksquare S \blacksquare D) = \partial Q_y \circledast \partial F_y \circledast \partial S_y \circledast \partial D_y,$$

where

$$\partial Q_y = \left[\begin{array}{c|c|c|c} Q_1 & Q_1 & Q_1 & \dfrac{\partial Q_1}{\partial q_1} \\ \hline Q_2 & Q_2 & Q_2 & \dfrac{\partial Q_2}{\partial q_2} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline Q_M & Q_M & Q_M & \dfrac{\partial Q_M}{\partial q_M} \end{array}\right], \quad \partial F_y = \left[\begin{array}{c|c|c|c} F_1 & F_1 & \dfrac{\partial F_1}{\partial x_1} & F_1 \\ \hline F_2 & F_2 & \dfrac{\partial F_2}{\partial x_2} & F_2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline F_M & F_M & \dfrac{\partial F_M}{\partial x_M} & F_M \end{array}\right]$$

$$\partial S_y = \left[\begin{array}{c|c|c|c} S_1 & \dfrac{\partial S_1}{\partial \omega_1} & S_1 & S_1 \\ \hline S_2 & \dfrac{\partial S_2}{\partial \omega_2} & S_2 & S_2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline S_M & \dfrac{\partial S_M}{\partial \omega_M} & S_M & S_M \end{array}\right], \quad \partial D_y = \left[\begin{array}{c|c|c|c} \dfrac{\partial F_1}{\partial x_1} & D_1 & D_1 & D_1 \\ \hline \dfrac{\partial F_2}{\partial x_2} & D_2 & D_2 & D_2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \dfrac{\partial F_M}{\partial x_M} & D_M & D_M & D_M \end{array}\right], \qquad (3)$$

while the symbol ⊙ denotes the transposition of the block end co-factor [1], which, by definition, is reduced to the block-by-block TEP operation, for instance,

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \circledcirc \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} \blacksquare B_{11} & A_{12} \blacksquare B_{12} \\ \hline A_{21} \blacksquare B_{21} & A_{22} \blacksquare B_{22} \end{array}\right].$$

the

Expression (3) provides a graphic illustration of the following feature of factorization of the derivative $\partial P / \partial Y$, which can be taken into account in the case of automatic formation:

1) all block-columns within the limits of the block matrix except for the block column of the derivatives are identical;

2) the block index in all block matrices corresponds to the column number of its elements different from zero (all other elements of the block are zero) and to the number of the source whose coordinate is used as an argument of the relevant characteristic, in particular, for S_4 in the matrix ∂S_y, given $M = 5$ we have

$$S_4 = \begin{bmatrix} 0 & 0 & 0 & S_1(\omega_4) & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & S_T(\omega_4) & 0 \end{bmatrix},$$

similarly

$$\frac{\partial S_1}{\partial \omega_1} = \begin{bmatrix} \frac{\partial S_1(\omega_1)}{\partial \omega_1} & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial S_1(\omega_1)}{\partial \omega_1} & 0 & 0 & 0 & 0 \end{bmatrix};$$

3) the block column of the derivatives for the inverse order of forming the vector of unknown Y moves in the block matrices from right to left (in the case under study

$$Y = \left[d_1 \ldots d_M \,\middle|\, \omega_1 \ldots \omega_M \,\middle|\, x_1 \ldots x_M \,\middle|\, q_1 \ldots q_M \right]^T);$$

4) the number of block columns in block matrices is determined by the number of the sources (for M sources — M block columns, while the number of the block columns is equal to the number of matrices participating in TEP formation).

This rather detailed consideration of the factorization properties of the Noidckkcr derivative plays an important role in the case of subsequent derivation of final relationships for composite blocks of the Fisher matrix whose composition includes the derivative $\partial P / \partial Y$. In particular for the DAA four-coordinate model, which has its conformity in matrix $P(1)$, the blocks of the Fisher matrix, which are outside the main diagonal may be expressed in the form

$$\left(\frac{\partial P}{\partial Y}\right)^T \cdot (A \otimes P) = [\partial Q_y^T \circledcirc \partial F_y^T \circledcirc \partial S_y^T \circledcirc \partial D_y^T] \cdot [A \otimes (Q \blacksquare F \blacksquare S \blacksquare D)], \quad (4)$$

$$(A^* \otimes P^T) \cdot \frac{\partial P}{\partial Y} = [A^* \otimes (Q^T \square F^T \square S^T \square D^T)] \times$$
$$\times [\partial Q_y \circledcirc \partial F_y \circledcirc \partial S_y \circledcirc \partial D_y], \quad (5)$$

where the symbol $\circledcirc$ denotes block end products [1] ($[A \circledcirc B]^T = = A^T \circledcirc B^T$).

As a result of investigations it was proved that relationships (4) and (5) are reduced to element-by-element Hadamard products (all relationships shown further on due to the extremely cumbersome intermediate computations are given without proof):

$$\left(\frac{\partial P}{\partial Y}\right)^T \cdot (A \otimes P) = [\partial Q_y^T \cdot (A \otimes Q)] \circ$$
$$\circ [\partial F_y^T \cdot (1_M \otimes F)] \circ [\partial S_y^T \cdot (1_M \otimes S)] \circ [\partial D_y^T \cdot (1_M \otimes D)],$$

$$(A^* \otimes P^T) \cdot \frac{\partial P}{\partial Y} \doteq [(A^* \otimes Q^T) \cdot \partial Q_y] \circ$$
$$\circ [(1_M^T \otimes F^T) \cdot \partial F_y] \circ [(1_M^T \otimes S^T) \cdot \partial S_y] \circ [(1_M^T \otimes D^T) \cdot \partial D_y],$$

where 1_M is the vector of unity amplitudes coinciding in terms of dimension with the vectors of amplitudes A (in the given case for M signals the M vector of units $1_M = [1 \ldots 1]^T$).

The lowering of the dimension of block expressions by reducing them to the Hadamard product makes it possible to substantially simplify both the reading of the resultant records of the Fisher matrix and the derivation of final relationships suitable for their numerical computations on a PC. This approach is especially efficient in the problems of analyzing the accuracy of multi-coordinate radars.

For the right-hand lower block of the Fisher matrix there is also an identity using the factorization of the derivative $\partial P / \partial Y$. One should note that the derivation of the final record for this block previously caused especially large difficulties due to the unwieldiness of necessary analytical calculations. From now on the given procedure will be noticeably simplified if one takes into account that

$$\left(\frac{\partial P}{\partial Y}\right)^T \cdot (A A^* \otimes 1_{RRTN}) \cdot \frac{\partial P}{\partial Y} =$$

$$= [\partial Q_y^T \circledcirc \partial F_y^T \circledcirc \partial S_y^T \circledcirc \partial D_y^T] \cdot [A A^* \otimes 1_R \otimes 1_R \otimes 1_T \otimes 1_N] \times$$
$$\times [\partial Q_y \circledast \partial F_y \circledast \partial s_y \circledast \partial D_y],$$

hence the sought-for identity, corresponding to (1), has the form

$$\left(\frac{\partial P}{\partial Y}\right)^T \cdot (A A^* \otimes 1_{RRTN}) \cdot \frac{\partial P}{\partial Y} =$$
$$= [\partial Q_y^T (A A^* \otimes 1_R) \partial Q_y] \circ [\partial F_y^T (1_{MM} \otimes 1_R) \partial F_y] \circ$$
$$\circ [\partial S_y^T (1_{MM} \otimes 1_T) \partial S_y] \circ [\partial D_y^T (1_{MM} \otimes 1_N) \partial D_y],$$

where 1_{MM} is the $M \times M$ matrix whose elements are equal to the unit.

Given the large dimension of the matrix P, for instance, when the number of the coordinates measured in the radar exceeds four, one should use the "adsorption" property of Kronecker products by the end ones:

$$[G^T \circledcirc N^T \circledcirc D^T \circledcirc L^T ...] \circ$$
$$\circ [A A^* \otimes 1_G \otimes 1_N \otimes 1_D \otimes 1_L ...] \circ [G \circledast N \circledast D \circledast L ...] =$$
$$= [G^T (A A^* \otimes 1_G) G] \circ [N^T (1_{MM} \otimes 1_N) N] \circ$$
$$\circ [D^T (1_{MM} \otimes 1_D) D] \circ [L^T (1_{MM} \otimes 1_L) L] ...$$

Finally, it remains for us to write down the simplest block of the Fisher matrix — the upper left one. Using the identity known from [4]

$$(A \square B)(C \blacksquare D) = (A C) \cdot (B D),$$

it is not difficult to obtain:

$$P^T \cdot P = (Q^T \square F^T \square S^T \square D^T) \cdot (Q \blacksquare F \blacksquare S \blacksquare D) =$$
$$= (Q^T \cdot Q) \circ (F^T \cdot F) \circ (S^T \cdot S) \circ (D^T \cdot D).$$

To find the Cramer–Rao lower bound the only remaining thing is to do the inversion of the Fisher matrix, which is sufficiently simple to do automatically in the MS Excel table processor to say nothing of the specialized Mathcad 7.0 (6.0) program.

A confirmation of the importance of results given is the properties of the limiting resolution of distance-measuring procedures [5] obtained on their basis.

It is expected that the approach set out here will be a basis for a theoretical breakthrough in solving the problem of the analytical estimation of the accuracy of multi-coordinate conformal multi-static radars with DAA functioning in the multi-signal mode. Unfortunately, the extreme growth of mathematical complexity of relevant computations limits their publication in the periodical literature.

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